

# Non-Hydrostatic Effects in the Interaction between Flow and Orography

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The interaction between flows and orography is a fundamental aspect of theoretical fluid dynamics for its direct applications (e.g., in dynamical meteorology); a comprehensive description is nowadays still lacking in some aspects. In this work, in particular, the authors would like to face the problem of flow-blocking and of the streamlines pattern formation, examining the role of stratification (i.e., Brunt-Vaisala frequency) and Froude number on these problems. In particular this work wants to investigate the role of vertical advection on flow-blocking and on streamlines geometry. The importance of streamlines curvature and stratification for the formation of pressure perturbation, then their role in flow-blocking will be shown. Moreover it will be shown how flow-blocking cannot be easily predict using only a stratification parameter or the Froude number.

## 1. Introduction

The interaction between flow and orography is an important topic of theoretical fluid dynamics because of its direct applications in everyday life. As an example, orographic rain is originated by a moist flow that, interacting with orography, gives rise to a vertical motion, then to condensation and precipitation formation. Even if these phenomena are very common, their explanation is nowadays not complete. Infact a full description requires the knowledge of the solution of Navier-Stokes equation with complex boundary conditions (i.e. top of troposphere and the orography).

The literature facing the interactions between flows and orography can be divided in: *i*) numerical works, *ii*) analytical works and *iii*) experimental works: Riley *et al.* (1976), Baines (1979), Hunt & Snyder (1980), Castro *et al.* (1983) and Snyder *et al.* (1985). Among the numerical studies there are several contributions produced using hydrostatic numerical models (Smolarkiewicz & Rotunno 1989) and more recently some contributions realized using non-hydrostatic numerical models (e.g., the Weather, Research and Forecasting model, WRF) (Miglietta & Rotunno 2005). The problem with numerical models, both hydrostatic and non-hydrostatic, is that their output is extremely complex then, generally, very difficult to interpret weighting the physical role of every possible parameter used in the tuning of the model (Giaiotti *et al.* 2007). On the contrary analytic works permit to keep a more complete control of the role of each parameter inserted into the analytical model even if some approximations need to be taken to reduce the mathematical difficulty of the starting equations.

Concerning the analytical works, several of them make use of the hydrostatic approximation and of different kind of obstacles (Lilly & Klemp 1979), using both stratified and rotational fluids (Inttlyre 1972), using thermal forcing (Reisner & Smolarkiewicz 1993) or imposing turbulent boundary layer at the surface (Carrunthers & Hunt 1990). Only a few analytical works avoid the use of the hydrostatic approximation, this because the vertical advective term makes the analytic approach more difficult. A comprehensive review of all these works can be found in Baines (1995).

Among the analytical works, three of them, Smith (1989*a*), Wurtele *et al.* (1987) and Keller (1994) deserve a special mention. In particular Smith (1989*a*), developing the previous work of Smith (1988) and Smith (1989*b*), studies the interaction between hydrostatic and stratified flow on an idealized 3-D topography. In his contribution the attention is focused on the case in which the flow stops its upward motion while moving on the topography (i.e., stagnation of the flow). Moreover, in this work, the use of Froude number as a discriminating factor between stagnation and non-stagnation, proposed by Sheppard (1956), is critically reviewed. In the work of Keller (1994) the study of interaction between non-hydrostatic and stratified flows on an idealized 2-D topography is presented but, in this case, the attention is focused on the effects of non-hydrostaticity on the formation of downstream lee waves. Keller (1994) also analyzes the behaviour of the flow with different vertical velocity profiles. The interesting aspect of Wurtele *et al.* (1987), instead, stays in the approach to the gravity wave propagation in stratosphere.

Following the line defined by the three analytical works above introduced, the aim of this work is to study analytically the influence of non-hydrostatic effects on the geometry of streamlines, essentially upstream to topography, and on flow blocking. In the § 2 a briefly review of Smith (1989*a*) is presented and its results are extended to the non-hydrostatic case. Then in § 3, an analytical model for a stratified flow on a 2-D profile is presented adopting a uniform incident velocity profile for the unperturbed flow and using the Fourier transforms. In this part the mathematical difficulty of antitransformation is evidenced and a new approach to overtake it is presented. With this approach the integrand is substituted with a new one for which the integration can be easily carried out. In this way, making use of the new integral form, the streamlines pattern for the non-hydrostatic case can be obtained and it is presented in § 4. With this approach the formation of lee waves, already reproduced by Keller (1994), as well as the intensification of wind speed at the top of topography profile is well described by the model. This phenomenon can be obtained even making use of hydrostatic models, but the non-hydrostatic approach shows a minor intensification of wind. Furthermore the changes in the pattern of streamline due to the super-critical (Froude number  $F > 1$ ) and sub-critical ( $F < 1$ ) regimes are presented. At the end of this work the relevance of non-hydrostatic effects are shown to be important in the dynamics of flow-blocking, § 5, specially for topographic profiles with horizontal scale comparable with the vertical one. Moreover, thanks to this study, stagnation can be put in relationship with the formation of vorticity as guessed before by Schar & Smith (1993*a*) and Schar & Smith (1993*b*). At the end it is demonstrated that simple parameters (e.g., Froude number or Brunt-Vaisala frequency) are not sufficient to describe completely the stagnation mechanism. This result is not merely theoretical but there are some concrete cases in which it might had played a role, as is the case of the Valcanale flood (29 August 2003, Valcanale-UD-, Italy) when two people died.

## 2. From hydrostatic to non-hydrostatic approach

In this section the work of Smith (1989) is adapted to the non-hydrostatic case. A 2-D steady and parallel flow with a constant vertical velocity profile is assumed. This flow

interacts with an obstacle whose analytic form is  $h = h(x, y)$ . Differently from Smith (1989a) a bounded flow is here considered. Infact, imposing that all the perturbation on pressure field and streamline displacement due to orography damp at infinite height can create problems to the energy conservation (Baines 1995). For this reason, in a more realistic way, we impose that all the orographic effects fade at the top of fluid that, in the atmospheric case coincides with the end of troposphere at 10 km.

As done by Smith (1989a), the flow is here assumed always parallel (i.e., turbulent diffusion is neglected), incompressible and stable stratified with constant Brunt-Vaisala frequency  $N$ . In this way far from orography the density profile  $\rho_\infty(z)$  is horizontally homogeneous and given by the following relationship

$$\rho_\infty(z) = \rho_0 \left( 1 - \frac{N^2(z - z_0)}{g} \right) \quad (2.1)$$

where  $z$  is the vertical coordinate,  $z_0$  is a reference level,  $\rho_0$  the density at the reference level far from the obstacles and  $g$  is the gravity acceleration.

The flow is assumed as composed by dry air in isothermal condition and all the sources and sinks of heat are neglected. The viscous effects and the turbulent diffusion of momentum are neglected as well. Moreover Coriolis force is assumed as null, infact Rossby number is lower than unity for an incident velocity profile of  $15 \text{ m s}^{-1}$  and for a horizontal scale smaller than 200 km.

To proceed further it is now necessary to introduce an energy conservation principle. For isentropic flows the energy for unit volume  $E$  given by

$$E = \frac{\rho U^2}{2} + P + \rho g z \quad (2.2)$$

is constant along stream and vortex lines (Batchelor 1994). In (2.3)  $P$  is the pressure and  $U$  is the intensity of velocity vector. In this case streamlines coincides with isopicnal lines because the flow is incompressible and steady. For this reason the energy balance written for a parcel on a streamline characterized by the unperturbed level  $z_0$  (i.e., far from the obstacle) is conserved and has the form

$$\frac{\rho_0 U_0^2}{2} + \rho_0 g z_0 + P_0 = \frac{\rho_0 U^2}{2} + \rho_0 g z + P$$

The pressure field far from orography is horizontally homogeneous as previously assumed, then it is possible to assume the hydrostatic balance to describe  $P_\infty$

$$\frac{\partial P_\infty(z)}{\partial z} = -\rho_\infty(z)g \quad (2.3)$$

The reasonableness of this assumption is given by the fact that no vertical motion, upstream and far from the obstacle, is assumed. Defining the perturbation pressure  $P^*$  as the difference between pressure field and pressure field at the same level and far from the obstacles

$$P^* = P - P_\infty$$

and using the equation (2.3), the velocity of a parcel along a streamline is given by

$$U^2 = U_0^2 - N^2 \eta^2 - 2 \frac{P^*}{\rho_0} \quad (2.4)$$

where  $U_0^2$  is the velocity of an incident upstream vertical profile while  $\eta$  is the streamline vertical displacement relative to the unperturbed streamline characterized by the height  $z_0$  far from the obstacle and density  $\rho_0$ .

The equation (2.4) gives us a simple relationship to identify the flow blocking, i.e. the situation in which  $U^2 = 0$ . Neglecting the perturbation pressure term, the results of Sheppard (1956) are reproduced. In those results stagnation occurs when the displacement is

$$F = \frac{U}{N\eta} < 1$$

The problem of this approach to flow-blocking is that neglecting pressure perturbation, the parcel behaves as if it were not immersed in a fluid environment. It is then important, for a more realistic description of the flow, try to estimate the pressure term. Starting from the vertical momentum equation (Emanuel 1994) for a stratified flow

$$\frac{Dw}{Dt} = -\frac{1}{\rho_\infty} \frac{\partial P^*}{\partial z} - \left( \frac{\rho'}{\rho_\infty} \right) g \quad (2.5)$$

where  $\rho'$  is the density perturbation, i.e. the difference between density field and density field at the same level and far from the obstacles, that is

$$\rho' = \rho(x, y, z) - \rho_\infty(z) = \frac{\rho_0 N^2 \eta}{g}$$

The result of integration of (2.5) from the general level  $z$  up to the top of the fluid  $D$  becomes

$$P^*(x, y, z, t) = \Gamma_h + \Gamma_{nh}$$

where

$$\Gamma_h = g \int_z^D \rho' dz \quad (2.6)$$

and

$$\Gamma_{nh} = \int_z^D \rho_\infty \frac{dw}{dt} dz \quad (2.7)$$

that correspond respectively to hydrostatic and non-hydrostatic contribution on the perturbation pressure

Using the isopical change of coordinates  $z = z_0 + \eta$  in the first integral  $\Gamma_h$ , equation (2.3) it becomes

$$U^2 = U_0^2 - 2N^2 I_\eta - 2 \frac{\Gamma_{nh}}{\rho_0} \quad (2.8)$$

where

$$I_\eta = \int_{z_0}^D \eta dz_0 \quad (2.9)$$

is the integral, obtained integrating streamlines displacement  $\eta$  from the unperturbed streamline level  $z_0$  to the top of the fluid where all streamlines perturbation are null.

In Smith (1989a) the non-hydrostatic contribution  $\Gamma_{nh}$  is neglected. This means that in equation (2.5) the vertical advective term is neglected. Using scale analysis and incompressibility condition it can be shown that the hydrostatic assumption is good only for obstacles with a vertical scale length smaller than the horizontal scale length. Using this approximation Smith (1989a) showed that stagnation occurs when the condition

$$F = \frac{U_0^2}{2N^2 I_\eta} < 1 \quad (2.10)$$

is satisfied. Stagnation dynamics is then connected to a non local variations of pressure.

In fact  $I_\eta$  is the vertical integral of all the vertical displacements above the fixed general streamline. The shape of the obstacle can become very important because the integral of streamlines vertical displacements depends from it. Moreover to determine stagnation all the streamlines displacement field  $\eta(x, y, z)$  have to be known. This means that the equations of motion (Navier-Stokes equations) had to be solved and a non linear solution of Navier-Stokes equation is not available. Then, to proceed further, a linear solution of the equation of motion has to be introduced to evaluate  $\eta$  and to insert it into the conservation of energy obtaining an evaluation of the stagnation point. There is then a formal contradiction: when stagnation occurs, streamlines become singular (e.g., they can split or intersect) then any linear solution is, in principle, not adapt to describe this behavior. To bypass this difficulty, in this work it is supposed that the linear solution, jointly with energy conservation, can give some information useful only to identify the onset of stagnation and not its behavior. <sup>†</sup> Even if this limitation is present, the result seem to be in agreement respect the numerical ones; for example Smith (1989b) has found that, introducing the values of streamlines displacement  $\eta(x, y, z)$  obtained by his 3-D linear hydrostatic models into (2.8), the critical hill height (i.e. the minimum hill height at which the flow starting to stop) is about 30% lower than the critical height prediction using the vertical displacement field obtained by a numerical integration of the full non-linear hydrostatic equation using in Smith (1989b). This fact suggest that the final resolution of the splitting problem will probably be accomplished using real observational data: this is not so easy because real data for stagnation position are difficult to obtain. Even though there are evident experimental difficulties, real data can give some hint on what in reality happens and what numerical models can be able to see respect analytical ones and viceversa. This paper, following the above idea, will present a comparison of result of analytical models on the case of Valcanale flood (29 August 2003, Valcanale-UD-, Italy).

Finally, It could be argued that if a linear theory is used to provide the field of motion, it would be more consistent to use the linearized set Bernoulli equation instead of the complete one (2.8). This could be done: Smith (1989b) found that the critical hill height is two times greater of the values prediction using the complete Bernoulli equation. Following Smith (1988), Smith (1989b) and Smith (1989a) we choose to use the exact result (2.8). Thus if any error is present in the derived fields it is because of the linearization leading to motion equation and not any subsequent linearizations and it is provides a common method for stagnation diagnosing (analytic) linear and (numerical) non-linear solution.

### 3. A non-hydrostatic model for the interaction between a flow and a 2-D topography

As shown in the previous section, the non-hydrostatic approach can become fundamental for obstacles characterized by horizontal scale comparable with the vertical one. In this case the vertical acceleration term plays an important role in the flow blocking phenomena and it has to be taken into account in the evaluation of the perturbation pressure term of (2.4).

To introduce the non-hydrostatic effects a 2-D model is developed and its governing

<sup>†</sup> It is possible, using other technique and approximation, taking in account part of the non-linear term and try to give some hint on behaviour of streamlines in the stagnation point. This aspect is not taking under consideration in this paper and further details can be found in Baines (1995).

equations will be solved explicitly in the super-critical (Froude number  $F > 1$ ) and sub-critical ( $F < 1$ ) regime. Then the streamlines pattern for both these cases and the flow-blocking dynamics will be described.

### 3.1. Integral representation of solutions

The development of the model starts from the governing equations of a 2-D incompressible, unviscid and stratified flow in Boussinesq's approximation (Emanuel 1994).

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho_\infty} \frac{\partial P^*}{\partial x} \\ \frac{Dw}{Dt} = -\frac{1}{\rho_\infty} \frac{\partial P^*}{\partial z} - \frac{g\rho'}{\rho_\infty} \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \end{cases} \quad (3.1)$$

where  $\rho_\infty$  is the unperturbed density field far from the obstacle. The set (3.1) is linearized imposing for the unperturbed state the horizontal and constant vertical velocity profile  $\mathbf{U} = (U_0, 0)$ . Moreover the perturbation velocity field  $\mathbf{u}'$ , using the streamfunction representation, can be described by

$$\mathbf{u}' = \left( -\frac{\partial \Psi}{\partial z}, \frac{\partial \Psi}{\partial x} \right) \quad (3.2)$$

It is then necessary to impose that all the perturbations damp at the fluid top while the linearized impermeability boundary condition at the ground according to Baines (1995) is satisfied, so

$$\begin{cases} \Psi = U_0 h(x) & \text{if } z = 0 \quad \text{ground} \\ \Psi = 0 & \text{if } z = D \quad \text{top of fluid} \end{cases} \quad (3.3)$$

In this way the perturbed streamfunction using a linear set of equations satisfies

$$\begin{cases} \nabla^2 \Psi - \frac{N^2}{g} \frac{\partial \Psi}{\partial z} + \frac{N^2}{U_0^2} \Psi = 0 \\ \Psi = 0 & \text{when } z = D \\ \Psi = U_0 h(x) & \text{when } z = 0 \end{cases} \quad (3.4)$$

This set of equations describes the behaviour of a 2-D stratified, steady, incompressible, parallel flow interacting with a general 2-D obstacle described by the shape  $z = h(x)$ . The linearity constraints the value of the parameters which determine the model. In particular the constraint is represented by the following two conditions on Brunt-Vaisala frequency  $N$

$$\begin{cases} \frac{NH}{U_0} \ll 1 \\ \frac{N^2 D}{g} \ll 1 \end{cases} \quad (3.5)$$

When the obstacle is represented by a 2-D hill of  $H = 2000$  m high and the flow is characterized by a upstream velocity  $U_0 = 15 \text{ m s}^{-1}$  the required stratification frequency that preserves linearity is  $N < 7.5 \cdot 10^{-3} \text{ s}^{-1}$ . This condition then limits the applicability of the above developed linear model. In the section §4 and §5 this constraint will be considered.

To proceed further the Fourier transform in reciprocal space  $k$  is here used to find an

integral representation of the solution. Assuming an orography at the ground with this shape

$$z = h(x) = \frac{H}{1 + (x/a)^2} \quad (3.6)$$

where  $a$  is the half width at half orography high. The solution has the form  $\Psi = \Psi_1 + \Psi_2$  where

$$\begin{cases} \Psi_1 = U_0 H a \int_0^c \cos(kx) e^{-ak} G_1 dk \\ \Psi_2 = U_0 H a \int_c^{+\infty} \cos(kx) e^{-ak} G_2 dk \end{cases} \quad (3.7)$$

with  $c = N/U_0$

$$G_1 = \frac{\sin((D - z)\lambda_1)}{\sin(D\lambda_1)} \quad (3.8)$$

$$G_2 = \frac{\sinh((D - z)\lambda_2)}{\sinh(D\lambda_2)} \quad (3.9)$$

and

$$\lambda_1 = \sqrt{c^2 - k^2} \quad (3.10)$$

$$\lambda_2 = \sqrt{k^2 - c^2} \quad (3.11)$$

### 3.2. Analytic integration

Obtaining an explicit analytic solution of (3.7) is not possible because the non-hydrostatic term is preserved in the strating equation (3.1). Moreover it can be noticed that, adopting the definition of Froude numeber given in Baines (1995)

$$F = \frac{\pi U_0}{ND} \quad (3.12)$$

two different behaviours exist, according to the different values of the Froude number.

When  $F > 1$  (super-critical regime) the functions  $G_1$  and  $G_2$  do not show any singularity on their integral path and the integral of (3.7) is defined. For the super-critical regime it is then possible to adopt a numerical integration of (3.7). In particular it is here adopted a “Monte-Carlo, importance sampling” integration with the use of an algorithm for the generation of pseudo-random numbers. This choice was taken because for low values of the wave number  $k$  and for low values of  $z$ , the integral function is nearly an harmonic function then, an equispatal integration method is not able to reproduce correctly the orographic profile near to the ground because it continuously adds harmonic components in phase. The use of a pseudo-random integration method, in which harmonic components are added randomly, seems a better approach. A comprehensive tractation of this integration method can be found in Gould & Tobochnik (1996).

In the case  $F < 1$  (sub-critical regime) the function  $G_1$  admits poles of first order for

$$k_n = \sqrt{c^2 - (n\frac{\pi}{D})^2} \quad \text{with } n \in \mathbb{N} \quad (3.13)$$

then the integral of (3.7) exists only in an improper way called “Cauchy principal value integral”. The presence of a pole in the integration path makes impossible the use of numerical integration methods and the analytic integration of the “Cauchy pricipal value integral” is needed to deal with the singularity behaviour but an explicit solution

of (3.7) is not available. To overcome this problem in this work a new approach is proposed based on the substitution of  $G_1$  and  $G_2$  with two new functions  $\widetilde{G}_1$  and  $\widetilde{G}_2$  such that

- The function  $\widetilde{G}_2$  has to tend to infinity with the same behaviour of  $G_2$ ; in this way there is a correct estimate of short wave lengths.
- The function  $\widetilde{G}_1$  has to reproduce the same singular behaviour and the limit of  $\widetilde{G}_1/G_1$  as  $k$  tends to the singularity has to be unitary.
- The function  $\widetilde{G}_1$  and  $\widetilde{G}_2$  have to have the same value in  $c = N/U_0$ .
- The function  $\widetilde{G}_1$  has to have the same value of  $G_1$  in  $k = 0$ ; this assures that the short wavenumber (i.e., wavelengths greater or equal to the horizontal scale of the obstacle) are correctly reproduced.
- With the new functions  $\widetilde{G}_1$  and  $\widetilde{G}_2$  the integral (3.7) can be solved explicitly.

At this point is necessary to fix some parameter of analytical model: with an upstream velocity  $U_0 = 15 \text{ m s}^{-1}$ , an hill high  $H = 2000 \text{ m}$  and a top of fluid due to the end of Troposphere  $D = 10 \text{ km}$ , linearization conditions (3.5) imposes that the stratification frequency must be  $N < 7.5 \cdot 10^{-3} \text{ s}^{-1}$ . So, notice the (3.13), there is only one pole  $k_0$

$$k_0 = \sqrt{c^2 - \left(\frac{\pi}{D}\right)^2} \quad (3.14)$$

In the following section §4 and §5 the above value of parameter are considered, so the function  $G_1$  have only one pole in the sub-critical case ( $F < 1$ ).

Following the previously introduced criteria, the new functions that are going to be uses instead of  $G_1$  and  $G_2$  become

$$\widetilde{G}_2 = \beta e^{-z(k-c)} \quad (3.15)$$

and

$$\widetilde{G}_1 = Mk + \alpha + W \frac{k(c-k)}{k-k_0} \quad (3.16)$$

where

$$\left\{ \begin{array}{l} M = \frac{\beta - \alpha}{c} \\ \alpha = \frac{\sin(c(D-z))}{\sin(Dc)} \\ \beta = 1 - z/D \\ \gamma = \sin(\beta\pi) \frac{\pi}{k_0^2 D^2} \\ P = \frac{\gamma}{c-k_0} \end{array} \right. \quad (3.17)$$

As we will be shown later on, the wave's dynamic is governed by the parameter  $W$ ; if  $W \neq 0$ , there is a wave pattern formation, if  $W = 0$  the wave pattern disappears.

Then, inserting the new functions in (3.7) it is possible to solve explicitly the integral (Gradshteyn & Ryzhik 2000), obtaining an explicit solution of  $\Psi$  for the sub-critical regime. It is important to notice that the different behaviour for  $F > 1$  and  $F < 1$  is a typical example of bifurcation, i.e., different physical behaviour as a consequence of an arbitrarily small change in the parameters value. The case  $F = 1$  can not be described by a simply linear theory infact, for this value of  $F$ , equation (3.7) shows a second order pole in  $k = 0$  and the integral does not exist, neither in the sense of “Cauchy principal value integral”.



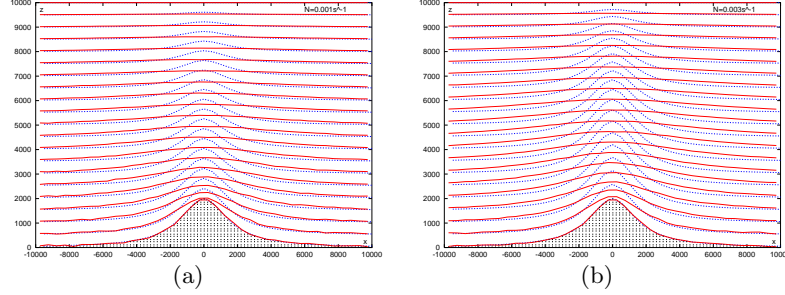


FIGURE 1. Super-critical regime of the flow with low stratification. Solid lines represent the streamlines of the non-hydrostatic model, while dashed lines represent the streamlines of the hydrostatic model. Horizontal and vertical axes scale are in meters. Panel (a) shows the streamlines pattern for  $N = 1 \cdot 10^{-3} \text{ s}^{-1}$  while panel (b) shows the results obtained for  $N = 3 \cdot 10^{-3} \text{ s}^{-1}$ .

#### 4. The comparison of the streamlines pattern in hydrostatic and non-hydrostatic approaches

In this section the streamlines pattern of the non-hydrostatic model are compared with those obtained under the hydrostatic assumption. This is done using the 2-D hydrostatic model proposed by Baines (1995) making use of the same assumptions (steady flow, unviscid, parallel, stratified and incompressible) here adopted.

$$\Psi = U_0 h(x) \frac{\sin(c(D - z))}{\sin(cD)} \quad (4.1)$$

From the linearization of boundary condition, fixing as a starting unperturbed level  $z = z_0$ , the streamline is then represented by  $z = z_0 + \Psi/U_0$ .

All the results presented in the following sections had been obtained adopting an hill-shaped obstacle whose functional form is (3.6) and with the geometrical parameters  $a = 1700 \text{ m}$  and  $H = 2000 \text{ m}$ . This value of hill parameter correspond to a characteristic narrow mountains of Alpine ridge. It is important to notice that, as it is told before, the topography must be narrow to have non-hydrostaticity importance.

##### 4.1. Super-critical regime ( $F > 1$ )

In figure 1 are displayed the hydrostatic and non-hydrostatic streamlines patterns. It is clear that the non-hydrostatic model is characterized by a lower vertical displacement respect the hydrostatic one at the same stratification frequency  $N$ . The origin of this behaviour is in the vertical acceleration term that the hydrostatic model can not take into account. Infact the increase of vertical streamlines displacement and velocity, due to the continuity equation as a consequence of narrowing of troposphere, gives rise to the formation of a low pressure zone at hill's top. This low pressure zone constrains the vertical streamlines displacement because the fluid in the upper part of troposphere is pushed downward by this pressure deficit. This lower pressure is more accentuated in the non-hydrostatic model because the extra term represented by the velocity vertical variation. Then streamlines displacement is reduced in the non-hydrostatic model by the action of this counteracting pressure gradient.

Before to conclude it has to be noted that the vertical displacement increases with the increasing of  $N$ . This behaviour can be observed both in the hydrostatic and non-hydrostatic model and might seem counter-intuitive (one could think that when stratification increases then the restoring forces become larger). This behavior is explained considering that the inertia of the lower-levels parcels increases more than the inertia of

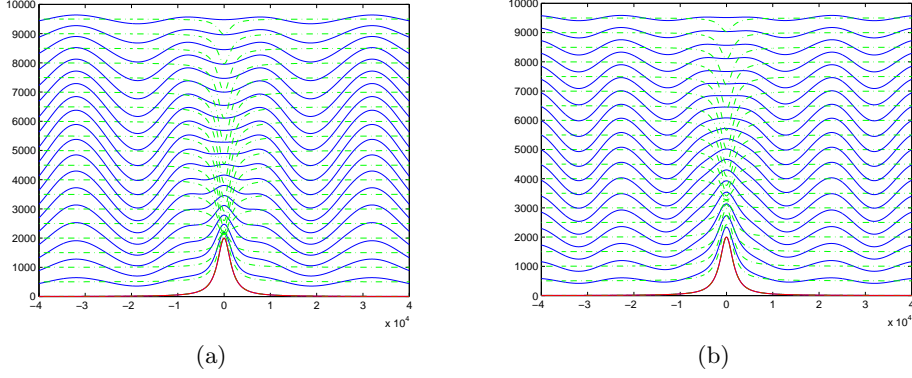


FIGURE 2. Sub-critical regime of the flow. Solid lines represent the streamlines for the non-hydrostatic model while dashed lines represent the streamlines for the hydrostatic model. Horizontal and vertical axes scale are in meters. Panel (a) shows the streamlines pattern for  $N = 6 \cdot 10^{-3} \text{ s}^{-1}$  while panel (b) shows the streamlines pattern for  $N = 7 \cdot 10^{-3} \text{ s}^{-1}$ .

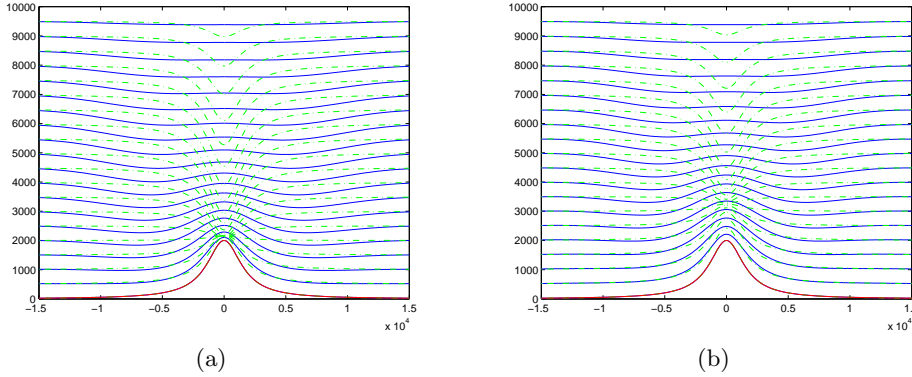


FIGURE 3. Sub-critical regime of the flow with high stratification. Solid lines represent the streamlines of the non-hydrostatic model while dashed lines represents the streamlines for the hydrostatic model. Horizontal and vertical axes scale are in meters. Panel (a) shows the streamlines pattern for  $N = 6 \cdot 10^{-3} \text{ s}^{-1}$  while panel (b) shows the streamlines pattern for  $N = 7 \cdot 10^{-3} \text{ s}^{-1}$ .

the upper-levels parcels when stratification increases. So, even if the restoring force becomes larger as stratification increase, the difference of masses of different air level parcel is crucial for the developing of perturbation pressure that play an important rule in the streamlines formation and, as it will be possible to see later, in flow-blocking dynamics.

#### 4.2. Sub-critical regime ( $F < 1$ )

Considering all the terms present in (3.16) the results shown in figure 2 are obtained. In that picture it is clearly recognizable the formation of stationary waves. These results are very similar to those found in Keller (1994). The formation of these waves is characteristic of the singularity, infact for a wind velocity of  $15 \text{ m s}^{-1}$  and a stratification frequency of the order of  $N = 7 \cdot 10^{-3} \text{ s}^{-1}$  there is a pole at

$$k_0 = \sqrt{c^2 - \left(\frac{\pi}{D}\right)^2} = 3.45 \cdot 10^{-4} \text{ m}^{-1} \quad (4.2)$$

which corresponds to a wavelength of  $\lambda \approx 18 \text{ km}$  that is clearly recognizable in figure 2.

Before to proceed further it has to be noted that the streamline waves pattern is symmetric to the top of the hill while from real cases it is clear that the waves pattern can be present only downstream to the hill. This is a spurious effect of the linearization, infact the (3.4) does not change for the reflection of the variable  $x$ . To avoid this spurious effect it is sufficient to fix  $W = 0$  in (3.16) and the streamlines pattern that one obtain is shown in figure 3. It is then interesting to notice that even if the waves are produced by the obstacle, their wavelength has not the same order of magnitude of the hill's horizontal scale. Infact, being triggered by the singular point  $k_0$ , the waves depend only from the stratification and from the velocity profile.

In figure 3 it is possible to clearly recognize the velocity intensification at the top of the obstacle represented by the streamlines concentration in that region. This intensification is a well known effect to mountain-hickers and it is due to the falling down of the upper-level streamlines which is not present in the super-critical stratification. This different behaviour comes out because in the sub-critical regime the upper-level parcels fall down in a denser environment that gives them the necessary up-ward lifting force to return to their initial level when they return far from obstacle. The obtained stationary pattern is represented in figure 3. This situation can not take place in the super-critical regime, when a downward displacement similar to that of the sub-critical case, cannot receive the same lifting force because of a less stratified environment.

It is interesting to notice that the non-hydrostatic model gives a lower intensification of wind speed at the top of the hill respect the hydrostatic ones at the same stratification frequency. A possible description of this effects can be explained, as was done in the super-critical case, considering the Bernoulli equation even if, in sub-critical case, the streamlines pattern is more complicated respect the super-critical one. In this case the acceleration of the fluid and the increase of vertical displacement of lower level streamlines gives the formation of a low pressure area at hill's top while the falling of upper streamlines could be enough to compensate the fluid acceleration and so this falling streamlines could give an increase of pressure: infact the mid-level streamlines, after an initial falling, seem to moves upward due to the acting of this increase of pressure as it is possible to see in figure 3. So, this more complicated pressure pattern gives rise to a lower intensification of wind speed at the top of the hill and, as it will be noted later, is the responsible of flow-blocking in sub-critical flow.

Finally it is worth to be noticed that, also in this case, the vertical displacement increases with the increasing of stratification (the altitude of the wind speed intensification increase grows with  $N$ ). The explanation given again makes use of the increasing of inertia with the increasing of stratification. Lower-level parcels moving upward find themselves immersed in a layer of fluid less dense, then with a low capability of counteracting with the buoyancy force their upward motion.

Before to conclude this section it is again important to state that, both in the super-critical and sub-critical regime, the non-hydrostatic effect consists in a smoothing of the streamlines geometry and the smoothing effect is concetreted near to the top of the obstacle.

## 5. Non-hydrostatic effects on flow-blocking

Knowing the streamfunction for the non-hydrostatic case, it is possible to can calculate the non-hydrostatic term  $\Gamma_{nh}$  of (2.7) due to the vertical acceleration and use it in (2.8). Four cases for different stratification frequencies  $N = 1, 3, 6, 7 \cdot 10^{-3} \text{ s}^{-1}$  are taken into account. The first two frequencies (i.e.,  $N = 1, 3 \cdot 10^{-3} \text{ s}^{-1}$ ) correspond to the super-critical regime while the last two (i.e.,  $N = 6, 7 \cdot 10^{-3} \text{ s}^{-1}$ ) to the sub-critical regime.

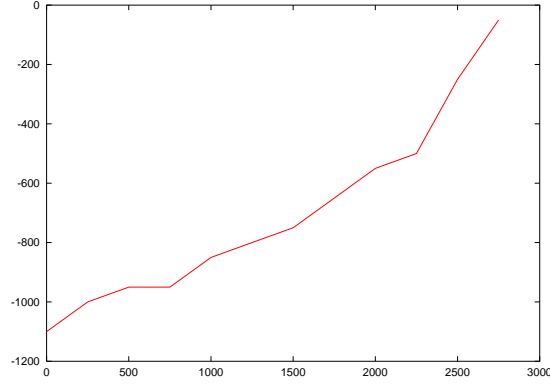


FIGURE 4. Flow-blocking and hydrostatic approximation. The unperturbed streamline level is reported in the horizontal axis, in meters, while the abscissa of stagnation (i.e. the distance from the top of the hill where the flow-blocking occurs) is reported, in meters, in the vertical axis. This picture is obtained with a stratification  $N = 3 \cdot 10^{-3} \text{ s}^{-1}$ .

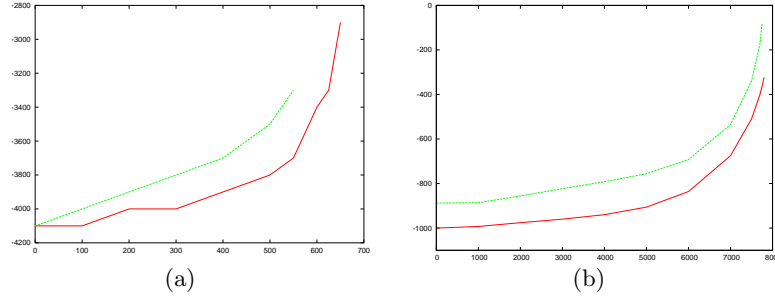


FIGURE 5. Non-hydrostatic case. In the horizontal axis it is reported the unpertubated streamline level (in meters) while the vertical axis reports, in meters, the abscissa of stagnation, i.e. the distance from the top of the hill where the fluid stops. Panel (a) reports the super-critical regime. Dash upper lines correspond to the stratification  $N = 3 \cdot 10^{-3} \text{ s}^{-1}$  while solid lower lines correspond to  $N = 1 \cdot 10^{-3} \text{ s}^{-1}$ . Panel (b) reports the sub-critical regime. Dash upper lines correspond to the stagnation  $N = 7 \cdot 10^{-3} \text{ s}^{-1}$  while solid lower lines to  $N = 6 \cdot 10^{-3} \text{ s}^{-1}$ .

### 5.1. Hydrostatic result

In the equation (2.8) the vertical advection term (2.7) is null because a hydrostatic model is here used, then the streamlines displacement is given by (4.1). The relationship between the unperturbed streamline level and stagnation abscissa is shown in figure 4. The only stratification frequency that admits stagnation is  $N = 3 \cdot 10^{-3} \text{ s}^{-1}$ , which corresponds to the streamlines pattern where the vertical displacement larger, as can be seen in figure 1. The explanation of this fact can be found in the behaviour of streamline pattern. In a super-critical hydrostatic flow, stagnation occurs when the streamlines does not have a sufficient kinetic energy to transform into potential to produce the vertical displacement request. On the contrary, the sub-critical flow blocking, as it will be remark in the next section, happens when the falling of upper streamlines creates a pressure configuration at the top of hill that could stop low-level streamlines

A last comment can be done on stagnation abscissa: there are streamlines, in the hydrostaic case, that stopped in the proximity of hill's top.

### 5.2. Non-Hydrostatic result

In the non-hydrostatic case flow stagnation can take place for all the considered stratification frequencies.

In the super-critical regime the behaviour of the abscissa of stagnation is shown in the panel *a* of figure 5. This behaviour is similar to that evidenced in the hydrostatic regime even if in the non-hydrostatic case the abscissa of stagnation is upward limited and does not pass the abscissa of 2500 m from the top of the hill. The explanation of this fact can be found in the role of the streamlines curvature. In figure 5a it is possible to observe that as all the stagnation abscissas are positioned before of the streamlines' flexums (for the ground streamline the flexum is at  $a \approx 1700$  km, value that corresponds to the flexum of (3.6)). This effect is similar to that experienced by a driver when he or she is running on the positive curvature of a road and his/her car is forced downward while on the negative curvature of the road the car is lifted. The same effect takes place in fluids but only a non-hydrostatic model can keep into account this mechanism because it is the only model that can consider vertical accelerations.

In the sub-critical regime the situation is quite different. The inertia of the lower-levels fluid parcels pushes the fluid beyond the flexum and stagnation occurs later than in the super-critical regime. The other important things to be noticed is that large portions of the fluid are blocked in the sub-critical regime (nearly the 8 km of fluid nearer to the ground). This situation corresponds to the pattern in figure 3 where the upper streamlines fall toward the obstacle. This means that large amounts of kinetic energy become potentially available. This fact is in agreement with the findings of Schar & Smith (1993a), Smolarkiewicz & Rotunno (1989) and, Castro *et al.* (1983) where stagnation is associated with the vorticity generation.

Concerning the role of stratification frequency on stagnation in figure 5 it can be shown that when  $N$  increases, stagnation occurs later, i.e., for smaller abscissas. This fact is in agreement with the streamlines pattern, in fact when stratification increases the vertical displacement is larger. This result might seem counterintuitive but it can be explained taking into account the fact that when stratification increases, restoring forces increase but the inertia of the lower parts of the fluid increases as well. This inertia is strictly connected to the generation of pressure perturbation, that play a crucial role in flow-blocking dynamics, and a non-hydrostatic model can better keep into account this fact in comparison with hydrostatic models. Even if this result is obtained with a linear model valid only for a restrict range of stratification frequencies it can give a useful hint toward the interpretation of real cases, in particular of what happened in Valcanale (UD), Italy, during the 29<sup>th</sup> August 2003.

In that day a flow strongly stratified in the lower levels ( $N = 3 \cdot 10^{-2} \text{ s}^{-1}$  compared to the mean value of  $N = 7 \cdot 10^{-3} \text{ s}^{-1}$ ) moving from south and interacting with the orographic ridge (Julian Preals and Alps) overcame the first ridge producing large amount of rain (nearly 400mm in four hours) and two casualties only on the further inner ridge (Julian Alps) where convection took place, as can be seen from the radar image shown in figure 6.

## 6. Conclusions

This paper presents a study on the interactions between stratified flows and orography carried out developing an analytical model. In particular, starting from the previous works of Smith (1989a) and Keller (1994), here the non-hydrostatic terms are kept into account in a simplified 2-D model to evaluate their effects on streamlines patterns and flow-

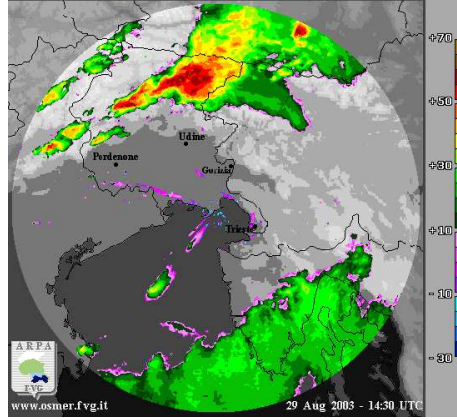


FIGURE 6. Radar image: a moist and stable stratified flow with  $N = 0.03 \text{ s}^{-1}$  interact with 2-D ridge. The flow overtaking the first ridge but stopped on the second (red and dark region) where starting convection and an intense orographic rains. Photo made by doppler radar station of Fossalon (Ud), Italy by Meteorological regional service O.S.M.E.R.

blocking. Following this idea an integral solution of the 2-D non-hydrostatic model had been found, whose behaviour is described by way of the Monte-Carlo sampling integration method for the super-critical regime and by way of a newly developed explicit integration approach in the sub-critical regime.

The main results of this work can be summarized in the following points:

- The non-hydrostatic effects are important for a topography characterized by a horizontal scale comparable with the vertical scale. In this case, in fact, the change in the topography curvature, then in the streamlines curvature that is connected to perturbative pressure, becomes important.
- The non-hydrostatic effects produces a general smoothing of the streamlines pattern. These effects can be explained taking into account the presence of the high/low pressure area, due to the complexity of streamlines picture, that a non-hydrostatic model are much able to see.
- The streamlines pattern shows, for the sub-critical regime, the formation of waves downstream to the topography, that hydrostatic models can not reproduce. The wave-length of this undulatory pattern is connected to the properties of a singular point in the integral function which depends only from the stratification frequency and from the upstream velocity of the flow. The intensification of flow velocity at the topography top is observed in the sub-critical regime.
- Stagnation occurs upstream to the topography and hydrostatic and non-hydrostatic models show a very different behavior in flow-blocking. In hydrostatic models stagnation occurs on the top of topography and only for large vertical displacements of streamlines (i.e, in the super-critical regime).
- In the non-hydrostatic model, for the super-critical regime, only lower-levels streamlines are blocked and stagnation does not occur beyond the flexum abscissa. This behaviour can be explained taking into account the role of streamline's curvature which is connected to the vertical perturbation pressure (lifting in the negative curvature part of streamlines and downward restoring force in the positive curvature part of streamlines) that can not be observed in hydrostatic models.
- In the non-hydrostatic model, for the sub-critical regime, the stagnation is due by a formation of complex pressure pattern at the top of hill. Moreover stagnation in sub-

critical case occurs beyond the flexum's abscissa upstream to topography. This because the inertia of the lower-levels parcels is larger than the inertia of the upper-levels parcels. In the sub-critical regime the amount of fluid that is blocked is very large: this means that a large amount of kinetic energy is made available. This source of kinetic energy might become a source of vorticity production connected with the flow-blocking as previously observed by Schar & Smith (1993*a*), Smolarkiewicz & Rotunno (1989) and Castro *et al.* (1983).

- As stratification increases, the flow stops later than a less stratified flow. For this reason stratification parameters alone are not sufficient to detect the flow-blocking situation because stratification and curvature effects are linked to the perturbation pressure that can have an important role on flow-blocking, as shown in this work. So this work suggest the opportunity to connect perturbation pressure to stratification and curvature parameters.

- This work suggest the possibility to construct a faster evaluation method of stagnation phenomena: the result of a linear model, that can be run faster and for a more smaller spatial mesh, could be use to describe the flow-blocking onset. This could be an important application for the wheater forecasting where the evaluation time is an important parameter to minimize in many circumstances.

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